

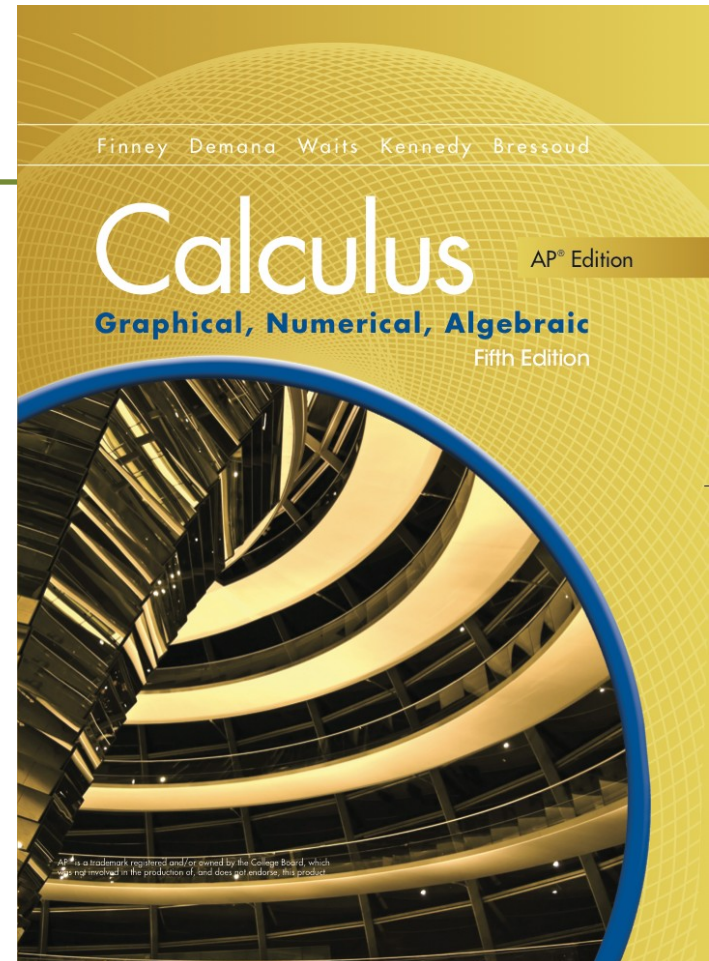
Chapter 9

Sequences, L'Hopital's Rule, and Improper Integrals



Section 9.1

Sequences



Quick Review

Let $f(x) = \frac{x}{x+4}$. Find the values of f .

1. $f(5)$
2. $f(-1)$

Evaluate the expression $a + (n - 1)d$ for the given values of a , n , and d .

3. $a = -$
 2 , $n = 2$, $d = 3$
4. $a = 1$, $n = -2$, $d = 2$

Quick Review

Evaluate the expression ar^{n-1} for the given values of a , r , and n

5. $a = \frac{1}{2}$, $r = 2$, $n = 3$

6. $a = -2$, $r = 1.5$, $n = 4$

Find the value of the limit.

7. $\lim_{x \rightarrow \infty} \frac{2x^2 - 2}{4x^2 + x - 1}$

8. $\lim_{x \rightarrow 0} \frac{\sin(4x)}{x}$

Quick Review Solutions

Let $f(x) = \frac{x}{x+4}$. Find the values of f .

1. $f(5)$ $\frac{5}{9}$

2. $f(-1)$ $-\frac{1}{3}$

Evaluate the expression $a + (n - 1)d$ for the given values of a , n , and d .

3. $a = -2$, $n = 2$, $d = 3$ 1

4. $a = 1$, $n = -2$, $d = 2$ -5

Quick Review Solutions

Evaluate the expression ar^{n-1} for the given values of a , r , and n .

5. $a = \frac{1}{2}$, $r = 2$, $n = 3$ 2

6. $a = -2$, $r = 1.5$, $n = 4$ - 6.75

Find the value of the limit.

7. $\lim_{x \rightarrow \infty} \frac{2x^2 - 2}{4x^2 + x - 1}$ $\frac{1}{2}$

8. $\lim_{x \rightarrow 0} \frac{\sin(4x)}{x}$ 4

What you'll learn about

- Explicit and recursive definitions
- Arithmetic and geometric sequences
- Graphs of sequences
- Limits of sequences; convergent and divergent sequences
- The Squeeze Theorem

...and why

Sequences arise frequently in mathematics and applied fields.

Defining a Sequence

A **sequence** $\{ a_n \}$ is a list of numbers written in an explicit order.

For example: $\{ a_n \} = \{ a_1, a_2, a_3, \dots, a_n, \dots \}$, where a_1 is the first **term** and a_n is the **nth term** of the sequence.

Let $\{ a_1, a_2, a_3, \dots, a_n, \dots \}$ be a function with domain the set of positive integers and range $\{ a_1, a_2, a_3, \dots, a_n, \dots \}$. If the domain is finite, then the sequence is a **finite sequence**. If the domain is infinite, then the sequence is an **infinite sequence**.

Example Defining a Sequence Explicitly

Find the first four terms and the 100th term of the sequence $\{a_n\}$

$$\text{where } a_n = \frac{(-1)^n}{n^2 + 2}.$$

Set n equal to 1, 2, 3, 4, and 100.

$$a_1 = \frac{(-1)^1}{1^2 + 2} = -\frac{1}{3}$$

$$a_2 = \frac{(-1)^2}{2^2 + 2} = \frac{1}{6}$$

$$a_3 = -\frac{1}{11}$$

$$a_4 = \frac{1}{18}$$

$$a_{100} = \frac{1}{10,002}$$

Example Defining a Sequence Recursively

Find the first three terms and the seventh term for the sequence defined recursively by the conditions: $b_1 = 4$ and $b_n = b_{n-1} - 2$ for all $n \geq 2$.

Since $b_1 = 4$ and $b_n = b_{n-1} - 2$, you can find $b_2 = 2$, $b_3 = 0$, and $b_7 = -8$.

Arithmetic Sequence

A sequence $\{a_n\}$ is an **arithmetic sequence** if it can be written in the form

$$\{a, a + d, a + 2d, \dots, a + (n - 1)d, \dots\}$$

for some constant d . The number d is the **common difference**

Each term in an arithmetic sequence can be obtained recursively from its preceding term by adding d :

$$a_n = a_{n-1} + d \text{ for all } n \geq 2.$$

Example Defining Arithmetic Sequences

Given the arithmetic sequence: - 3, 1, 5, 9, ... find

- (a) the common difference,
- (b) the ninth term,
- (c) a recursive rule for the n th term,
- (d) an explicit rule for the n th term.

(a) The common difference between terms is 4.

(b) $a_9 = -3 + (9 - 1)(4) = 29.$

(c) $a_1 = -3$ and $a_n = a_{n-1} + 4$ for all $n \geq 2$.

(d) $a_n = -3 + (n - 1)(4)$
 $= 4n - 7$

Geometric Sequence

A sequence $\{a_n\}$ is a **geometric sequence** if it can be written in the form

$$\{a, a \cdot r, a \cdot r^2, \dots, a \cdot r^{n-1}, \dots\}$$

for some nonzero constant r . The number r is the **common ratio**.

Each term in a geometric sequence can be obtained recursively from its preceding term by multiplying by r :

$$a_n = a_{n-1} \cdot r \text{ for all } n \geq 2.$$

Example Defining Geometric Sequences

For the geometric sequence 1, - 3, 9, - 27, ..., find

- (a) the common ratio,
- (b) the tenth term,
- (c) a recursive rule for the n th term,
- (d) an explicit rule for the n th term.

(a) The common ratio is - 3.

(b) $a_{10} = (1) \cdot (-3)^9 = -19683$

(c) The sequence is defined recursively by $a_1 = 1$ and $a_n = (-3) a_{n-1}$ for $n \geq 2$.

(d) The sequence is defined explicitly by $a_n = (1) (-3)^{n-1} = (-3)^{n-1}$.

Example Constructing a Sequence

The second and fifth term of a geometric sequence are - 6 and 48, respectively. Find the first term, common ratio and an explicit rule for the n th term.

$$\frac{a_1 r^4}{a_1 r} = - \frac{48}{6}$$

$$r^3 = - 8$$

$$r = - 2$$

Then $a_1 \cdot r = - 6$ means that $a_1 = 3$.

The sequence is defined explicitly: $a_n = (3)(- 2)^{n-1} = (- 1)^{n-1} (3)(2)^{n-1}$

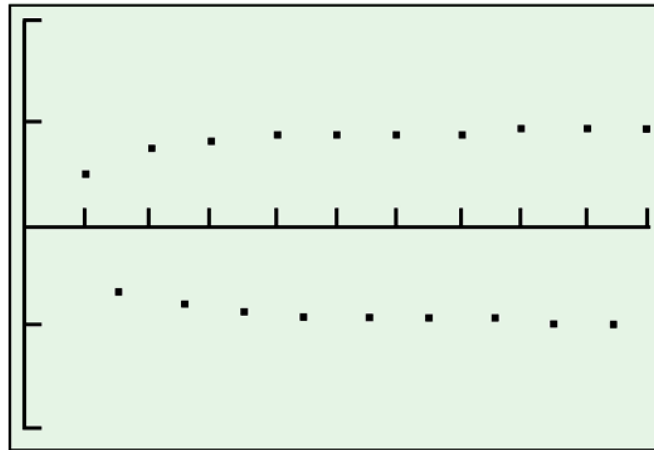
Example Graphing a Sequence Using Parametric Mode

Draw a graph of the sequence $\{a_n\}$ with $a_n = (-1)^n \frac{n-1}{n}$, $n=1, 2, \dots$

Let $X_{1T} = T$, $Y_{1T} = (-1)^T \frac{T-1}{T}$, and graph in dot mode. Set $T_{\min} = 1$,

$T_{\max} = 20$, and $T_{\text{step}} = 1$. Choose $X_{\min} = 0$, $X_{\max} = 20$, $X_{\text{scl}} = 2$, $Y_{\min} = -2$,

$Y_{\max} = 2$, $Y_{\text{scl}} = 1$, and draw the graph

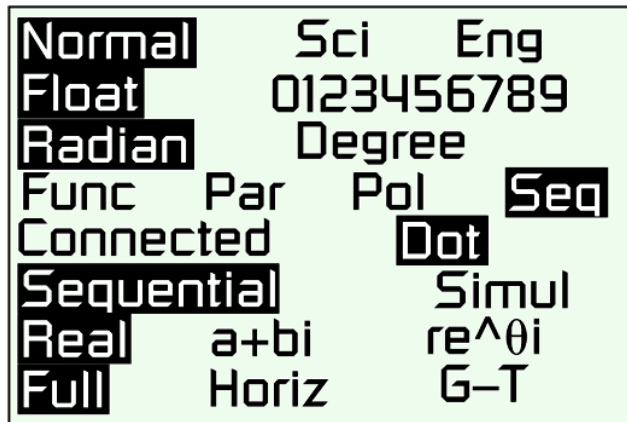


$[0, 20]$ by $[-2, 2]$

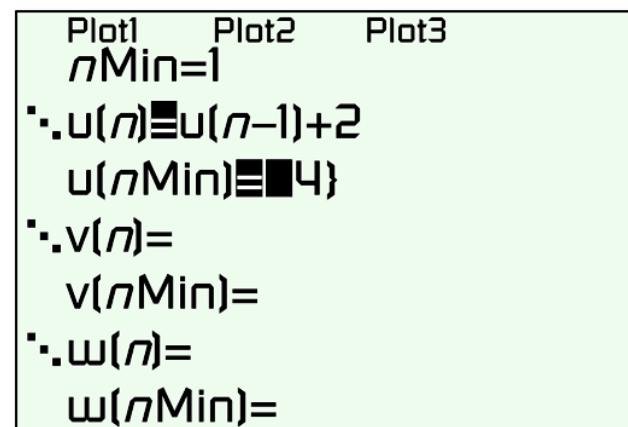
Example Graphing a Sequence Using Sequence Graphing Mode

Graph the sequence defined recursively by $b_1 = 4$ and
 $b_n = b_{n-1} + 2$ for all $n \geq 2$.

Set the graph in sequence graphing mode and dot mode.
 Replace b_n by $u(n)$. Select $n\text{Min} = 1$, $u(n) = u(n-1) + 2$,
 and $u(n\text{Min}) = \{ 4 \}$.



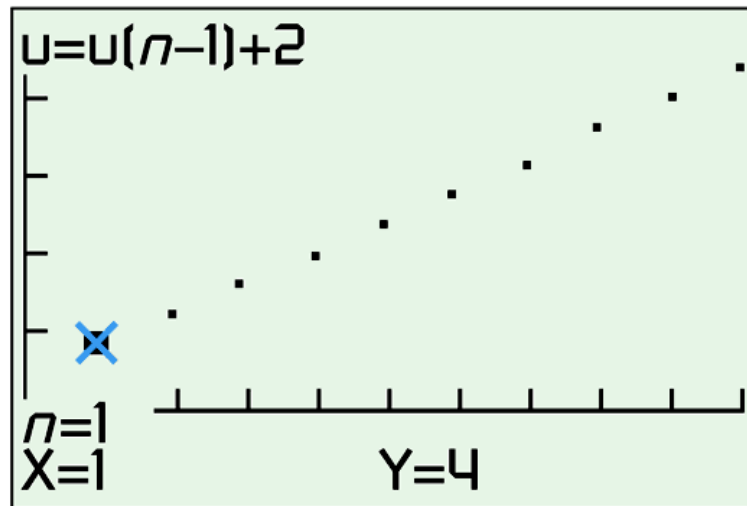
(a)



(b)

Example Graphing a Sequence Using Sequence Graphing Mode

Set $n\text{Min} = 1$, $n\text{Max} = 10$, $\text{PlotStart} = 1$, $\text{PlotStep} = 1$, and graph in the $[0, 10]$ by $[-5, 25]$ viewing window



$[0, 10]$ by $[-5, 25]$

Limit of a Sequence

Let L be a real number. The sequence a_n has **limit L as n approaches ∞** if we can force a_n to be as close to L as we wish for all sufficiently large n . More formally, given any positive number ε , there is a positive number M such that for all $n > M$ we have

$$|a_n - L| < \varepsilon.$$

We write $\lim_{n \rightarrow \infty} a_n = L$ and say that the sequence **converges to L** .

Sequences that do not have limits **diverge**.

Properties of Limits

If L and M are real numbers and $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} b_n = M$, then

1. Sum Rule: $\lim_{n \rightarrow \infty} (a_n + b_n) = L + M$

2. Difference Rule: $\lim_{n \rightarrow \infty} (a_n - b_n) = L - M$

3. Product Rule: $\lim_{n \rightarrow \infty} (a_n b_n) = L \cdot M$

4. Constant Multiple Rule: $\lim_{n \rightarrow \infty} (c a_n) = c \cdot L$

5. Quotient Rule: $\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = \frac{L}{M}, M \neq 0$

Example Finding the Limit of a Sequence

Determine whether the sequence converges or diverges. If it converges,

find its limit. $a_n = \frac{2n-1}{n}$

Analytically, using the Properties of Limits:

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{2n-1}{n} &= \lim_{n \rightarrow \infty} \left(2 - \frac{1}{n} \right) \\ &= \lim_{n \rightarrow \infty} (2) - \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) \\ &= 2 - 0 = 2\end{aligned}$$

The Squeeze Theorem for Sequences

If $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ and if there is an integer N for which $a_n \leq b_n \leq c_n$ for all $n > N$, then $\lim_{n \rightarrow \infty} b_n = L$.

Absolute Value Theorem

Consider the sequence $\{a_n\}$. If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.